Uniqueness results for the Minkowski problem extended to hedgehogs

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by

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Abstract.- *The classical Minkowski problem has a natural extension to hedgehogs, that is to Minkowski differences of closed convex hypersurfaces. This extended Minkowski problem is much more difficult since it essentially boils down to the question of solutions of certain Monge-Ampère equations of mixed type on the unit sphere* $\mathbb{S}^n$ *of* $\mathbb{R}^{n+1}$. *In this paper, we mainly consider the uniqueness question and give first results.*

0. Introduction

The classical Minkowski problem is that of the existence, uniqueness, and regularity of closed convex hypersurfaces of the Euclidean linear space $\mathbb{R}^{n+1}$ whose Gauss curvature (in the sense of Gauss' definition) is prescribed as a function of the outer normals. In the last century, this fundamental problem played an important role in the development of the theory of elliptic Monge-Ampère equations. Indeed, for C_+^2-hypersurfaces (C^2-hypersurfaces with positive Gauss curvature), this Minkowski problem is equivalent to the question of solutions of certain Monge-Ampère equations of elliptic type on the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$.

Using approximation by convex polyhedra, Minkowski proved the existence of a weak solution [17]: If K is a continuous positive function on $\mathbb{S}^n$ satisfying the following integral condition

$$\int_{\mathbb{S}^n} \frac{u}{K(u)}\, d\sigma(u) = 0,$$

where σ is the spherical Lebesgue measure on $\mathbb{S}^n$, then K is the Gauss curvature of a unique (up to translation) closed convex hypersurface $\mathcal{H}$. The uniqueness comes from the equality condition in a Minkowski's inequality (e.g. [21, p. 397]). The strong solution is due to Pogorelov [20] and Cheng and Yau [4] who proved independently that: if K is of class $C^m(\mathbb{S}^n;\mathbb{R})$, ($m \geq 3$), then the support function of $\mathcal{H}$ is of class $C^{m+1,\alpha}$ for every $\alpha \in \,]0,1[$.

This classical Minkowski problem has a natural extension to hedgehogs, that is to Minkowski differences $\mathcal{H} = \mathcal{K} - \mathcal{L}$ of closed convex hypersurfaces $\mathcal{K}, \mathcal{L} \in \mathbb{R}^{n+1}$, at least if we restrict ourselves to hypersurfaces whose support functions are C^2. Indeed, the inverse of the Gauss curvature of such a hedgehog is well defined and continuous all over $\mathbb{S}^n$ (see Section 1), so that the following existence question arises naturally:

(Q_1) *Existence of a C^2-solution: What necessary and sufficient conditions must a real continuous function $R \in C(\mathbb{S}^n; \mathbb{R})$ satisfy to be the curvature function (that is, the inverse $\frac{1}{K}$ of the Gauss curvature K) of some hedgehog $\mathcal{H} = \mathcal{K} - \mathcal{L}$?*

Now let us expound the uniqueness question. As we shall see later, for any $h \in C^2(\mathbb{S}^2; \mathbb{R})$, $-h$ and h are the respective support functions of two hedgehogs $\mathcal{H}_{-h}$ and $\mathcal{H}_h$ of $\mathbb{R}^3$ that have the same curvature function and are such that

$$\mathcal{H}_{-h} = s(\mathcal{H}_h),$$

where s is the symmetry with respect to the origin of $\mathbb{R}^3$. Here, we have to recall that noncongruent hedgehogs of $\mathbb{R}^3$ may have the same curvature function [16]: for instance, the two smooth (but not analytic) functions f, g defined on $\mathbb{S}^2$ by

$$f(u) := \begin{cases} \exp(-1/z^2) & \text{if } z \neq 0 \\ 0 & \text{if } \quad z = 0 \end{cases} \quad \text{and} \quad g(u) := \begin{cases} sign(z)\, f(u) & \text{if } z \neq 0 \\ 0 & \text{if } \quad z = 0, \end{cases}$$

where $u = (x, y, z) \in \mathbb{S}^2$, are the support functions of two noncongruent hedgehogs $\mathcal{H}_f$ and $\mathcal{H}_g$ of $\mathbb{R}^3$ having the same curvature function $R := 1/K \in C(\mathbb{S}^2; \mathbb{R})$, (cf. Figure 1).

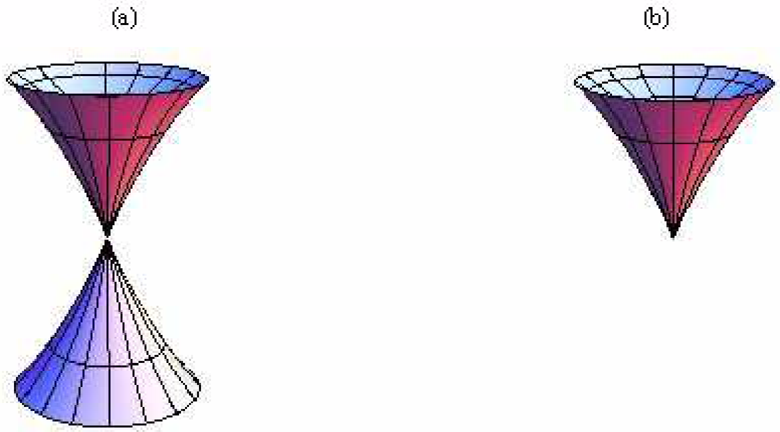

Figure 1. Noncongruent hedgehogs with the same curvature function

Consequently, we state the uniqueness question as follows:

(Q_2) *Uniqueness of a C^2-solution: If $R \in C(S^n; R)$ is the curvature function of some hedgehog $\mathcal{H}$, what necessary and sufficient additional conditions must* *it satisfy in order that $\mathcal{H}$ be the unique hedgehog of which R is the curvature function (up to an isometry of the space) ?*

In particular, it would be very interesting to know whether there exists any pair of noncongruent analytic hedgehogs of $\mathbb{R}^3$ with the same curvature function (by 'analytic hedgehogs' we mean 'hedgehogs with an analytic support function'). We shall see in Section 1 that this latest question presents similarities to the open question of knowing whether there exists any pair of isometric noncongruent analytic closed surfaces in $\mathbb{R}^3$.

For $n = 1$, the problem is linear and so can be solved without difficulty [15]. But for $n = 2$, the problem is already very difficult : if $R \in C(\mathbb{S}^2; \mathbb{R})$ changes sign on $\mathbb{S}^2$, the question of existence, uniqueness and regularity of a hedgehog of which R is the curvature function boils down to the study of a Monge-Ampère equation of mixed type, a class of equations for which there is no global result but only local ones by Lin [8] and Zuily [23]. In the present paper, we are mainly interested in the uniqueness question. Question (Q_2) is too difficult to be solved at the present time and our main purpose will be simply to provide conditions under which two hedgehogs of $\mathbb{R}^3$ must have distinct curvature functions.

As we shall recall in Section 2, any hedgehog can be uniquely split into the sum of its centered and projective parts. Let H_3 denote the $\mathbb{R}$-linear space of C^2-hedgehogs defined up to a translation in $\mathbb{R}^3$. One of our main results is the following.

Theorem. *Let $\mathcal{H}$ and $\mathcal{H}'$ be two C^2-hedgehogs that are linearly independant in H_3 and the centered parts of which are nontrivial (i.e., distinct from a point) and proportional to a same convex surface of class C^2_+. Then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.*

An immediate consequence will be that:

Corollary. *Two C^2-hedgehogs of nonzero constant width that are linearly independant in H_3 have distinct curvature function.*

In Section 1, we shall begin by recalling some basic definitions and facts and by presenting what is already known on the Minkowski problem extended to hedgehogs. Later, we shall see different ways of constructing pairs of noncongruent hedgehogs having the same curvature functions.

Section 2 will be devoted to the statement of the main results and Section 3 to the proofs and further remarks.

1. Basic facts and observations on the extended Minkowski problem

As is well-known, every convex body $K \subset \mathbb{R}^{n+1}$ is determined by its support function $h_K : \mathbb{S}^n \longrightarrow \mathbb{R}$, where $h_K(u)$ is defined by $h_K(u) = \sup\{\langle x, u\rangle \,|x \in K\}$, $(u \in \mathbb{S}^n)$, that is, as the signed distance from the origin to the support hyperplane with normal u. In particular, every closed convex hypersurface of class C^2_+ (i.e., C^2-hypersurface with positive Gauss curvature) is determined by its support function h (which must be of class C^2 on $\mathbb{S}^n$ [21, p. 111]) as the envelope $\mathcal{H}_h$ of the family of hyperplanes with equation $\langle x, u\rangle = h(u)$. This envelope $\mathcal{H}_h$ is described analytically by the two following equations

$$\begin{cases} \langle x, u\rangle = h(u) \\ \langle x, .\rangle = dh_u(.) \end{cases},$$

of which the second is obtained from the first by performing a partial differentiation with respect to u. From the first equation , the orthogonal projection of x onto the line spanned by u is $h(u)\,u$ and from the second one, the orthogonal projection of x onto $u^\perp$ is the gradient of h at u (cf. Figure 2). Therefore, for each $u \in \mathbb{S}^n$, $x_h(u) = h(u)u + (\nabla h)(u)$ is the unique solution of this system.

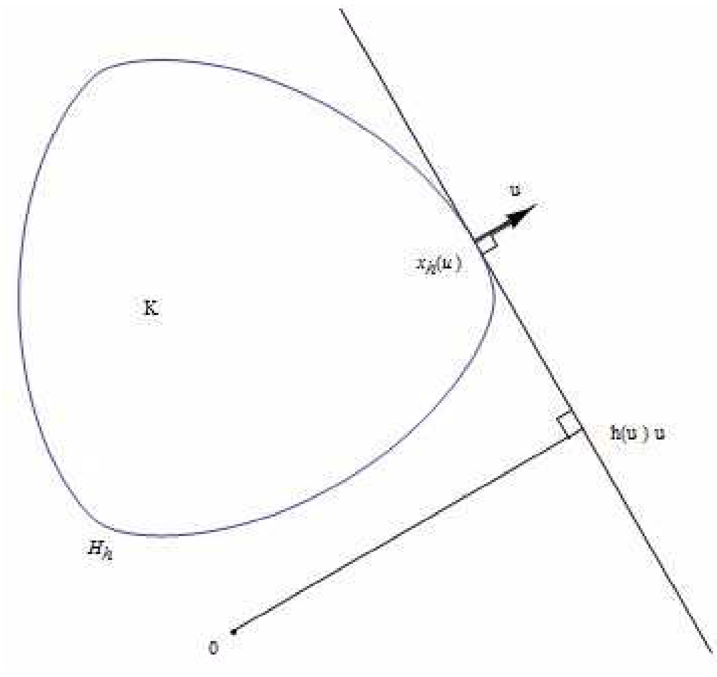

Figure 2. Hedgehogs as envelopes parametrized by their Gauss map

Now, for any C^2-function h on $\mathbb{S}^n$, the envelope $\mathcal{H}_h$ is in fact well-defined (even if h is not the support function of a convex hypersurface). Its natural parametrization $x_h : \mathbb{S}^n \to \mathcal{H}_h, u \mapsto h(u)u + (\nabla h)(u)$ can always be interpreted as the inverse of its Gauss map, in the sense that: at each regular point $x_h(u)$ of $\mathcal{H}_h$, u is normal to $\mathcal{H}_h$. We say that $\mathcal{H}_h$ is the hedgehog with support function h (cf. Figure 3). Note that x_h depends linearly on h.

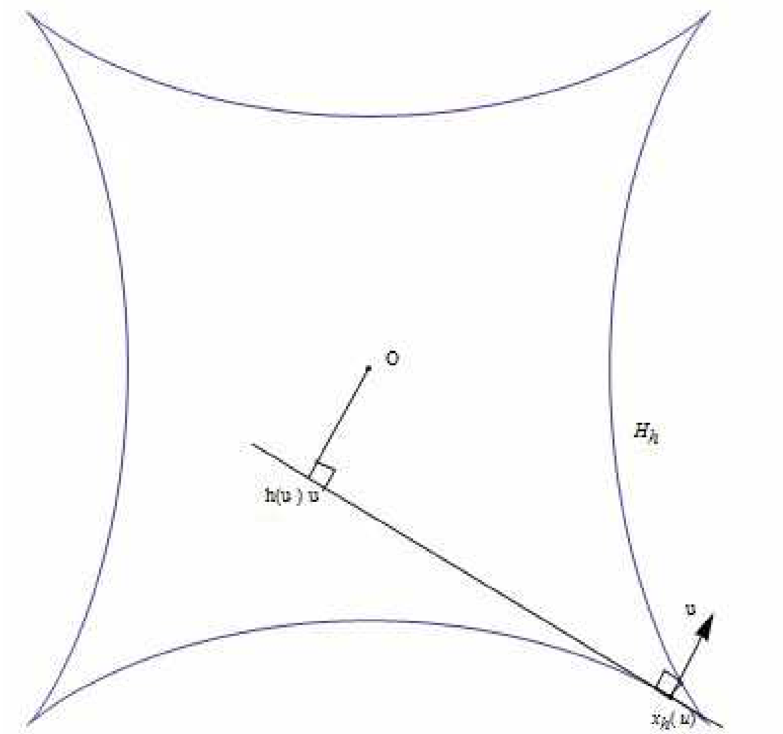

Figure 3. A hedgehog with a C^2-support function

Hedgehogs with a C^2-support function can be regarded as the Minkowski differences of convex hypersurfaces (or convex bodies) of class C^2_+. Indeed [7], given any $h \in C^2(\mathbb{S}^2; \mathbb{R})$, for all large enough real constant r, $h + r$ and r are support functions of convex hypersurfaces of class C^2_+ such that $h = (h + r) - r$.

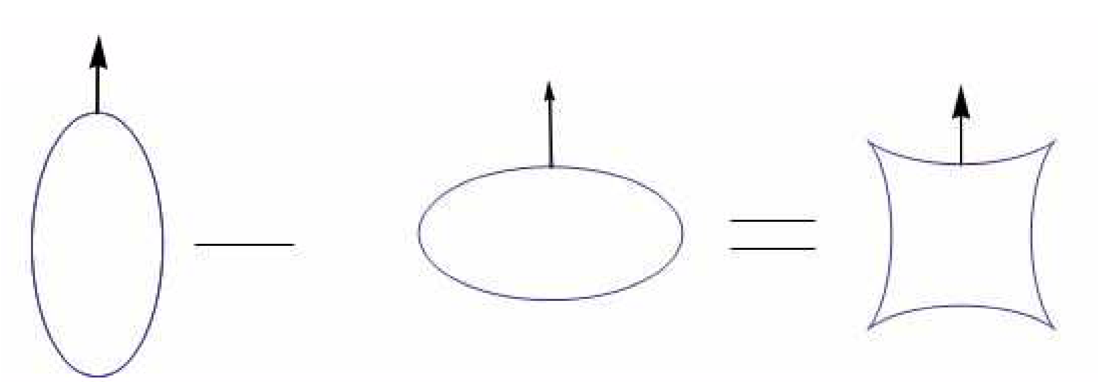

Figure 4. Hedgehogs as differences of convex bodies of class C^2_+

In fact, we can introduce a more general notion of hedgehogs regarding hedgehogs of $\mathbb{R}^{n+1}$ as Minkowski differences of arbitrary convex bodies of $\mathbb{R}^{n+1}$ [15]. For instance, Figure 5 represents a polygonal hedgehog obtained by subtracting two squares in $\mathbb{R}^2$. For $n \leq 2$, the idea of considering each formal difference of convex bodies of $\mathbb{R}^{n+1}$ as a (possibly singular and self-intersecting) hypersurface of $\mathbb{R}^{n+1}$ goes back to a paper by H. Geppert [4] who introduced hedgehogs under the German names *stützbare Bereiche* ($n = 1$) and *stützbare Flächen* ($n = 2$).

In the present paper, we shall only consider hedgehogs with a C^2-support function and we will refer to them as 'C^2-hedgehogs'.

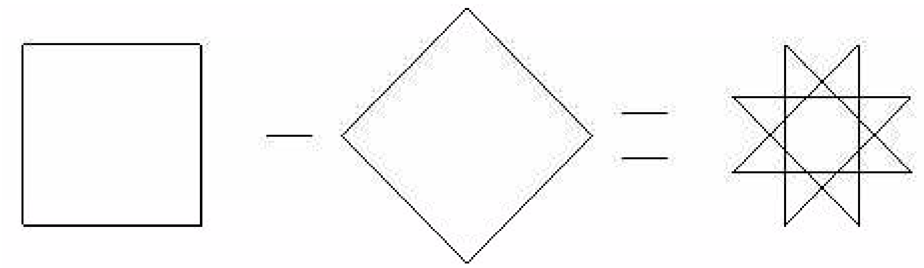

Figure 5. Hedgehogs as differences of arbitrary convex bodies

Gauss curvature of C^2-hedgehogs

Let H_{n+1} denote the $\mathbb{R}$-linear space of C^2-hedgehogs defined up to a translation in the Euclidean linear space $\mathbb{R}^{n+1}$ and identified with their support functions. Analytically speaking, saying that a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ is defined up to a translation simply means that the first spherical harmonics of its support funtion is not precised.

As we saw before, elements of H_{n+1} may be singular hypersurfaces. Since the parametrization x_h can be regarded as the inverse of the Gauss map, the Gauss curvature K_h of $\mathcal{H}_h$ is given by 1 over the determinant of the tangent map of x_h: $\forall u \in \mathbb{S}^n$, $K_h(u) = 1/\det[T_u x_h]$. Therefore, singularities are the very points where the Gauss curvature is infinite. For every $u \in \mathbb{S}^n$, the tangent map of x_h at the point u is $T_u x_h = h(u)\, Id_{T_u\mathbb{S}^n} + H_h(u)$, where $H_h(u)$ is the symmetric endomorphism associated with the hessian of h at u. Consequently, if λ is an eigenvalue of the hessian of h at u then $\lambda + h(u)$ is (up to the sign) one of the principal radii of curvature of $\mathcal{H}_h$ at $x_h(u)$ and the so-called 'curvature function' $R_h := 1/K_h$ can be given by

$$R_h(u) = \det\left[H_{ij}(u) + h(u)\,\delta_{ij}\right], \tag{1}$$

where δ_{ij} are the Kronecker symbols and $(H_{ij}(u))$ the Hessian of h at u with respect to an orthonormal frame on $\mathbb{S}^n$.

Case where $n = 2$. From (1), the curvature function $R_h := 1/K_h$ of $\mathcal{H}_h \subset \mathbb{R}^3$ is given by $R_h = (\lambda_1 + h)(\lambda_2 + h) = h^2 + h\Delta_2 h + \Delta_{22} h$, where Δ_2 denotes the spherical Laplacian and Δ_{22} the Monge-Ampère operator (respectively the sum and the product of the eigenvalues λ_1, λ_2 of the Hessian of h). So, the equation we shall be dealing with will be the following

$$h^2 + h\Delta_2 h + \Delta_{22} h = 1/K.$$

Note that the so-called 'mixed curvature function' of hedgehogs of $\mathbb{R}^3$, that is,

$$\begin{aligned} R : \mathsf{H}_3^2 &\to C\left(\mathbb{S}^2; \mathbb{R}\right) \\ (f, g) &\mapsto R_{(f,g)} := \tfrac{1}{2}\left(R_{f+g} - R_f - R_g\right) \end{aligned}$$

is bilinear and symmetric:

$(i)\, \forall\, (f, g, h) \in \mathsf{H}_3^3, \forall \lambda \in \mathbb{R}, R_{(f+\lambda g,h)} = R_{(f,h)} + \lambda R_{(g,h)}$;

$(ii)\, \forall\, (f, g) \in \mathsf{H}_3^2, R_{(g,f)} = R_{(f,g)}$.

For any $h \in \mathsf{H}_3$, we have in particular $R_{-h} = R_h$. Note that $R_{(1,f)} = \frac{1}{2}\left(\Delta_2 h + 2h\right)$ is (up to the sign) half the sum of the principal radii of curvature of $\mathcal{H}_h \subset \mathbb{R}^3$.

The Minkowski problem for hedgehogs

The point is that the curvature function $R_h := 1/K_h$ of any C^2-hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is well-defined and continuous all over $\mathbb{S}^n$, including at the singular points of x_h, so that the Minkowski problem arises naturally for hedgehogs. In this paper, we are thus interested in studying the existence and/or uniqueness of C^2-solutions to the Monge-Ampère equation

$$R_h = R, \tag{2}$$

where R is a given real continuous function on $\mathbb{S}^n$.

As in the classical Minkowski problem, the following integral condition is necessary for the existence of such a solution:

$$\int_{\mathbb{S}^n} R\left(u\right) u d\sigma(u) = 0. \tag{3}$$

It simply expresses that any C^2-hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is a closed surface. But it is no longer sufficient: for instance, the constant function equal to -1 on $\mathbb{S}^2$ satisfies integral condition (3) but it cannot be the curvature function of a hedgehog since there is no compact surface with negative Gauss curvature in $\mathbb{R}^3$.

Examples where equation (2) **is of mixed type and has no C^2-solution under condition** (3)

This extended Minkowski problem leads to the following examples of Monge Ampère equations of mixed type for which there is no solution. For every $v \in \mathbb{S}^2$, the smooth function $F_v(u) = 1 - 2\langle u, v\rangle^2$ satisfies integral condition (3) but is not a curvature function on $\mathbb{S}^2$ [13]. In other words, for every fixed $v \in \mathbb{S}^2$, the Monge Ampère equation $h^2 + h\Delta_2 h + \Delta_{22} h = F_v$ does not admit a solution on $\mathbb{S}^2$. The proof makes use of orthogonal projection techniques adapted to hedgehogs.

Non-uniqueness in the extended Minkowski problem

As recalled in the introduction, two noncongruent hedgehogs of $\mathbb{R}^3$ may have the same curvature function. By bilinearity and symmetry in the arguments of the mixed curvature function $R : \mathsf{H}_3^2 \to C(\mathbb{S}^2; \mathbb{R})$, if $\mathcal{H}_f$ and $\mathcal{H}_g$ are two hedgehogs of $\mathbb{R}^3$ having the same curvature function then, for all $(\lambda, \mu) \in \mathbb{R}^2$, the hedgehogs $\mathcal{H}_{\lambda f+\mu g}$ and $\mathcal{H}_{\mu f+\lambda g}$ also have the same curvature funtion. For instance, from the pair $\{\mathcal{H}_f, \mathcal{H}_g\}$ of non-isometric hedgehogs represented in Figure 1, we deduce the pair $\{\mathcal{H}_{f+2g}, \mathcal{H}_{2f+g}\}$ of noncongruent hedgehogs (which have the same curvature function) represented in Figure 6.

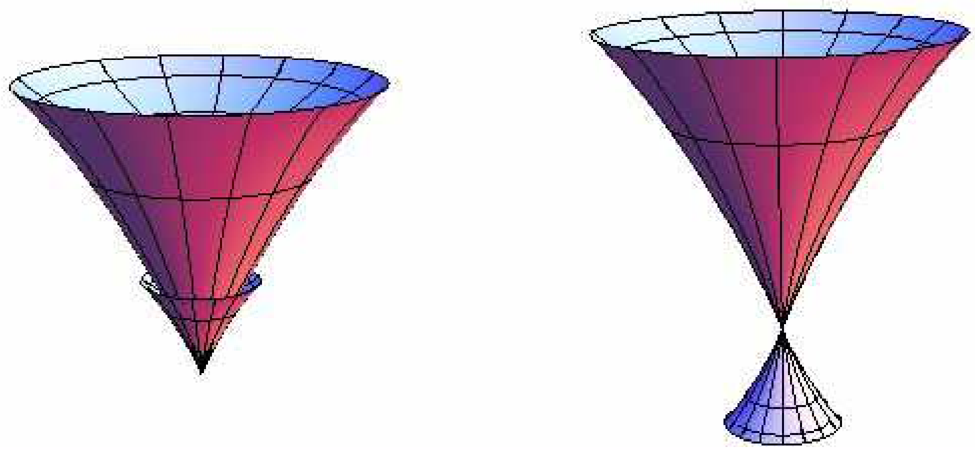

Figure 6. Noncongruent hedgehogs with the same curvature function

A natural but probably difficult question is knowing whether there exists any pair of noncongruent analytic hedgehogs of $\mathbb{R}^3$ with the same curvature function. Let us recall the similar open question of knowing whether there exists any pair of noncongruent isometric analytic closed surfaces in $\mathbb{R}^3$. Smooth closed surfaces can be isometric without being congruent : the usual way of constructing such surfaces is by gluing together smaller congruent pieces. As recalled in [3, p. 131] or [22, p. 366], we can for instance construct a pair $\{S, S'\}$ of noncongruent isometric closed surfaces of revolution as indicated in Figure 7.

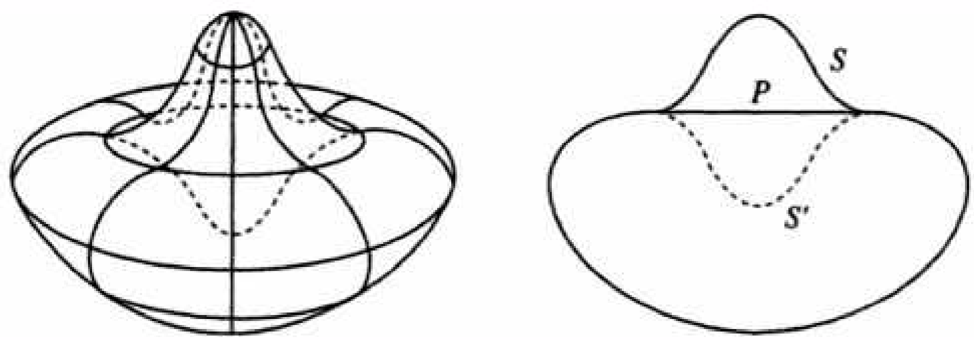

Figure 7. Noncongruent isometric surfaces of revolution [3, p. 131]

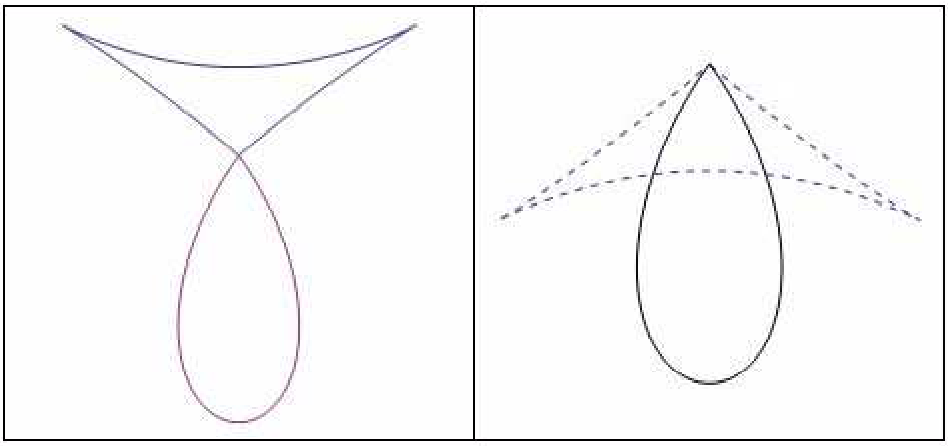

Figure 8. Generatrices of revolution hedgehogs with the same curvature

Obviously, we can place the origin at such a point of the axis of revolution that S admits a parametrization of the form

$$\begin{aligned} x : \mathbb{S}^2 &\rightarrow S \subset \mathbb{R}^3 \\ u &\mapsto \rho(u)\, u, \end{aligned}$$

where ρ is a smooth positive function. Then the hedgehog with support funtion $h = 1/\rho$ can be regarded as the dual surface of S [9]. This hedgehog $\mathcal{H}_h$ is a surface of revolution whose generating curve (a plane hedgehog which has a fish form) is represented in Figure 8. Replacing the fish's tail by its image under the symmetry with respect to the double point (which by duality corresponds to the plane P) and rotating the plane hedgehog that we get around its axis of symmetry, we generate an other hedgehog which has the same curvature function as $\mathcal{H}_h$ without being congruent to it.

2. Statement of results

Recall that H_3 denotes the $\mathbb{R}$-linear space of C^2-hedgehogs defined up to a translation in $\mathbb{R}^3$. The following result will be obtained as a consequence of the classical Minkowski's uniqueness theorem.

Theorem 1. *Let $\mathcal{H}_f$ and $\mathcal{H}_g$ be C^2-hedgehogs that are linearly independent in H_3. If the plane spanned by $\mathcal{H}_f$ and $\mathcal{H}_g$ in H_3 contains some hypersurface of class C^2_+, then $\mathcal{H}_f$ and $\mathcal{H}_g$ have distinct curvature functions.*

The second result makes use of the decomposition of hedgehogs into their centered and projective parts.

Decomposition of a hedgehog into its centered and projective parts

Recall that a hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is said to be centered (resp. projective) if its support function h is symmetric (resp. antisymmetric), that is, if we have:

$$\forall u \in \mathbb{S}^n, \quad h(-u) = h(u) \quad (\text{resp.} \quad h(-u) = -h(u)).$$

For instance, the hedgehog $\mathcal{H}_f$ (resp. $\mathcal{H}_g$) of $\mathbb{R}^3$ that is represented in Figure 1.a (resp. Figure 1.b) is centered (resp. projective). Geometrically speaking, saying that $\mathcal{H}_h$ is centered (resp. projective) means that $\mathcal{H}_h$ is centrally symmetric with respect to the origin (resp. that any pair of antipodal points on the unit sphere $\mathbb{S}^n$ correspond to a same point on the hypersurface $\mathcal{H}_h = x_h(\mathbb{S}^n)$).

Now, the support function h of $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ can be uniquely split into the sum of its symmetric and antisymmetric parts f and g:

$$\forall u \in \mathbb{S}^n, \quad h(u) = f(u) + g(u) \quad \text{where} \quad \begin{cases} f(u) = \frac{1}{2}(h(u) + h(-u)) \\ \\ g(u) = \frac{1}{2}(h(u) - h(-u)) \end{cases}.$$

Consequently, any hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ has a unique representation of the form $\mathcal{H}_f + \mathcal{H}_g$, where $\mathcal{H}_f$ and $\mathcal{H}_g$ are respectively a centered and a projective hedgehog. We say that $\mathcal{H}_f$ and $\mathcal{H}_g$ are respectively the centered and the projective part of $\mathcal{H}_h$.

Theorem 2. *Let $\mathcal{H}_{h_1}$ and $\mathcal{H}_{h_2}$ be C^2-hedgehogs that are linearly independent in H_3 and the centered parts of which are nontrivial (i.e., distinct from a point) and proportional to a same convex surface of class C^2_+. Then $\mathcal{H}_{h_1}$ and $\mathcal{H}_{h_2}$ have distinct curvature functions.*

A hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is said to be of constant width if its centered part has a constant support function. In other words, a hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is of constant width if the signed distance between the two cooriented support hyperplanes that are orthogonal to $u \in \mathbb{S}^n$ does not depend on u, that is, if:

$$\exists r \in \mathbb{R}, \forall u \in \mathbb{S}^n, h(u) + h(-u) = 2r.$$

A straightforward consequence of Theorem 3 is the following corollary.

Corollary 3. *Let $\mathcal{H}_f$ and $\mathcal{H}_g$ be C^2-hedgehogs that are linearly independent in H_3. If $\mathcal{H}_f$ and $\mathcal{H}_g$ are of nonzero constant width, then their support functions R_f and R_g are distinct.*

A hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is said to be analytic if its support function h is C^ω on $\mathbb{S}^n$.

Theorem 4. *Let $\mathcal{H}_f$ and $\mathcal{H}_g$ be analytic (resp. projective) C^2-hedgehogs of $\mathbb{R}^3$ that are linearly independent in H_3. If the mixed curvature function of $\mathcal{H}_f$ and $\mathcal{H}_g$ does not change sign on $\mathbb{S}^2$, then $\mathcal{H}_f$ and $\mathcal{H}_g$ have distinct curvature functions.*

3. Proof of the results

Proof of Theorem 1. By assumption, there exists $(\lambda, \mu) \in \mathbb{R}^2$ such that the hedgehog $\mathcal{H}_{\lambda f+\mu g}$ is of class C^2_+. Even if it means replacing (λ, μ) by $(\lambda + \delta, \mu)$ for any sufficiently small $\delta > 0$, we can assume that $|\lambda| \neq |\mu|$.

Let assume that $R_f = R_g$. We then have:

$$\begin{aligned} R_{\lambda f+\mu g} &= \lambda^2 R_f + \mu^2 R_g + 2\lambda\mu R_{(f,g)} \\ &= \mu^2 R_f + \lambda^2 R_g + 2\mu\lambda R_{(f,g)} \\ &= R_{\mu f+\lambda g}. \end{aligned}$$

As the hedgehog $\mathcal{H}_{\lambda f+\mu g}$ is of class C^2_+, the equality $R_{\lambda f+\mu g} = R_{\mu f+\lambda g}$ implies the existence of an $\varepsilon \in \{-1, 1\}$ such that $\lambda f + \mu g = \varepsilon (\mu f + \lambda g)$ by Minkowski's uniqueness theorem. We thus have $(\lambda - \varepsilon\mu) f = \varepsilon (\lambda - \varepsilon\mu) g$ and hence $f = \varepsilon g$ since $\lambda - \varepsilon\mu \neq 0$. $\square$

Lemma 5. *Let $\mathcal{H}_f$ and $\mathcal{H}_g$ be two C^2-hedgehogs of $\mathbb{R}^3$. If $u \in \mathbb{S}^2$ is such that $R_g(u) > 0$, then*

$$R_{(f,g)}(u)^2 \geq R_f(u)\, R_g(u)\,.$$

Proof of Lemma. Define $Q : \mathbb{R} \to \mathbb{R}$ by $Q(t) = R_{f+tg}(u)$. By bilinearity and symmetry of the mixed curvature function, we have:

$$\forall t \in \mathbb{R}, \quad Q(t) = R_f(u) + 2tR_{(f,g)}(u) + t^2 R_g(u)\,.$$

Let us consider the discriminant $\Delta = R_{(f,g)}(u)^2 - R_f(u)\, R_g(u)$ of this quadratic trinomial $Q(t)$. From the assumption $R_g(u) > 0$, it follows that $Q(t) > 0$ for any large enough t. On the other hand, there exists some $\lambda \in \mathbb{R}$ such that $R_{(1,f+\lambda g)}(u) = R_{(1,f)}(u) + \lambda R_{(1,g)}(u) = 0$ and hence $Q(\lambda) = R_{f+\lambda g}(u) \leq 0$. Therefore $\Delta \geq 0$, which achieves the proof. □

Surprisingly, there exist nontrivial (i.e., distinct from a point) hedgehogs of $\mathbb{R}^3$ whose curvature function is nonpositive all over $\mathbb{S}^2$ [12, 19], which disproves a conjectured characterization of the 2-sphere [1, 6]. However, the support function of such a hedgehog cannot be analytic or antisymmetric on $\mathbb{S}^2$ (in the analytic case the first proof is due to A.D. Alexandrov [2]):

Lemma 6 ([10, *Theorem 3*]). *Let $\mathcal{H}_h$ be an analytic (resp. a projective) hedgehog in $\mathbb{R}^3$. If the curvature function R_h of $\mathcal{H}_h$ is nonpositive all over $\mathbb{S}^2$, then $\mathcal{H}_h$ is reduced to a single point.*

Lemma 7. *Let $\mathcal{H}_g$ be a convex hedgehog of class C^2_+ in $\mathbb{R}^3$. Given a projective hedgehog $\mathcal{H}_f$ in $\mathbb{R}^3$, the mixed curvature function $R_{(f,g)}$ is equal to zero on $\mathbb{S}^2$ only if $\mathcal{H}_f$ is reduced to a single point, that is, only if f is the restriction to $\mathbb{S}^2$ of a linear form on $\mathbb{R}^3$.*

Proof of Lemma. Since $\mathcal{H}_g$ is of class C^2_+, we have

$$R_f(u)\, R_g(u) \leq R_{(f,g)}(u)^2$$

by Lemma 5. From $R_{(f,g)}(u) = 0$, we then deduce that $R_f \leq 0$ which implies the result by Lemma 6. □

Proof of Theorem 2. By assumption, h_1 and h_2 are of the form

$$\begin{cases} h_1 = f_1 + \lambda_1 g \\ \\ h_2 = f_2 + \lambda_2 g \end{cases},$$

where λ_1, λ_2 are nonzero real numbers and g the support function of a convex surface of class C^2_+. Assume that $R_{h_1} = R_{h_2}$. By bilinearity and symmetry of the mixed curvature function, this gives

$$R_{f_1} + \lambda_1^2 R_g + 2\lambda_1 R_{(f_1,g)} = R_{f_2} + \lambda_2^2 R_g + 2\lambda_2 R_{(f_2,g)}.$$

Splitting into symmetric and antisymmetric parts, we get

$$\begin{cases} R_{f_1} + \lambda_1^2 R_g = R_{f_2} + \lambda_2^2 R_g \\ \\ \lambda_1 R_{(f_1,g)} = \lambda_2 R_{(f_2,g)} \end{cases}.$$

By linearity of the mixed curvature function in the first argument, the second equation is equivalent to $R_{(\lambda_1 f_1 - \lambda_2 f_2, g)} = 0$. From Lemma 7, this implies that $\mathcal{H}_{\lambda_1 f_1 - \lambda_2 f_2}$ is a point and hence that $\mathcal{H}_{\lambda_1 f_1} = \mathcal{H}_{\lambda_2 f_2}$ in H_3. Now, by multiplying each member of the first equation of the previous system by λ_1^2, we get

$$\lambda_1^2 R_{f_1} + \lambda_1^4 R_g = \lambda_1^2 R_{f_2} + \lambda_1^2 \lambda_2^2 R_g,$$

and hence

$$R_{\lambda_1 f_1} - R_{\lambda_1 f_2} = \lambda_1^2 \left(\lambda_1^2 - \lambda_2^2\right) R_g$$

by bilinearity of the mixed curvature function. Therefore,

$$R_{\lambda_2 f_2} - R_{\lambda_1 f_2} = \lambda_1^2 \left(\lambda_1^2 - \lambda_2^2\right) R_g,$$

that is,

$$\left(\lambda_2^2 - \lambda_1^2\right)\left(R_{f_2} - \lambda_1^2 R_g\right) = 0.$$

As $\mathcal{H}_{f_2}$ is projective (resp. $\mathcal{H}_g$ is convex of class C^2_+), we have [10]:

$$\int_{\mathbb{S}^2} R_{f_2} d\sigma \leq 0 \quad \text{and} \quad \int_{\mathbb{S}^2} R_g d\sigma > 0.$$

Therefore, $R_{f_2} \neq \lambda_1^2 R_g$. From the previous equation, we thus get $\lambda_2^2 = \lambda_1^2$, that is:

$$\exists \varepsilon \in \{-1, 1\}, \lambda_2 = \varepsilon \lambda_1.$$

Now, $\lambda_1 \mathcal{H}_{f_1} = \mathcal{H}_{\lambda_1 f_1} = \mathcal{H}_{\lambda_2 f_2} = \lambda_2 \mathcal{H}_{f_2}$ and λ_1, λ_2 are nonzero. Therefore, we have $\mathcal{H}_{f_1} = \varepsilon \mathcal{H}_{f_2}$ in H_3, that is, $\mathcal{H}_{f_2} = \varepsilon \mathcal{H}_{f_1}$ and hence

$$\mathcal{H}_{h_2} = \mathcal{H}_{f_2 + \lambda_2 g} = \mathcal{H}_{f_2} + \lambda_2 \mathcal{H}_g = \varepsilon\left(\mathcal{H}_{f_1} + \lambda_1 \mathcal{H}_g\right) = \varepsilon \mathcal{H}_{h_1} \text{ in } \mathsf{H}_3,$$

which contradicts the fact that $\mathcal{H}_{h_1}$ and $\mathcal{H}_{h_2}$ are linearly independent in H_3. □

Lemma 8. *Let $\mathcal{H}_f$ and $\mathcal{H}_g$ be two C^2-hedgehogs of $\mathbb{R}^3$. If their curvature functions R_f and R_g are identically equal on $\mathbb{S}^2$, then $R_{f-g}(u) \leq 0$ or $R_{f+g}(u) \leq 0$ for all $u \in \mathbb{S}^2$.*

Proof of Lemma. Assume that $R_{f-g}(u) > 0$ (resp. $R_{f+g}(u) > 0$). By Lemma 5, we then have

$$R_{(f-g,f+g)}(u)^2 \geq R_{f-g}(u)\, R_{f+g}(u)\,.$$

Now the assumption $R_f = R_g$ implies

$$R_{(f-g,f+g)} = R_f - R_g = 0 \quad \text{and hence} \quad R_{f-g}(u)\, R_{f+g}(u) \leq 0.$$

Therefore $R_{f-g}(u) \leq 0$ (resp. $R_{f+g}(u) \leq 0$). □

Proof of Theorem 4. Let us prove the contrapositive. Assume that R_f and R_g are identically equal on $\mathbb{S}^2$. Since $\mathcal{H}_f$ and $\mathcal{H}_g$ are analytic (resp. projective and C^2) hedgehogs of $\mathbb{R}^3$ that are linearly independent in H_3, it follows from Lemma 6 that there must exist $(u,v) \in \mathbb{S}^2 \times \mathbb{S}^2$ such that $R_{f-g}(u) > 0$ and $R_{f+g}(v) > 0$. By Lemma 8, we then deduce that $R_{f+g}(u) \leq 0$ and $R_{f-g}(v) \leq 0$. Now we have $R_{(f,g)} = \frac{1}{4}(R_{f+g} - R_{f-g})$, so that

$$\begin{cases} R_{f-g}(u) > 0 \\ \\ R_{f+g}(u) \leq 0 \end{cases} \quad \text{and} \quad \begin{cases} R_{f+g}(v) > 0 \\ \\ R_{f-g}(v) \leq 0 \end{cases}$$

implies $R_{(f,g)} < 0$ and $R_{(f,g)}(v) > 0$. □

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